diagonalize the matrices in exercises 7 20 if possible

Diagonalize the matrices in exercises 7 and 20 if possible. Diagonalization is a crucial concept in linear algebra that allows us to simplify matrix operations and gain deeper insights into the properties of linear transformations. By expressing a matrix in diagonal form, we can easily compute powers of matrices, solve differential equations, and understand the underlying geometry of transformations. In this article, we will explore the process of diagonalizing matrices, specifically focusing on exercises 7 and 20, discussing necessary conditions, methods, and providing detailed examples.

Understanding Diagonalization

Diagonalization of a matrix involves finding a diagonal matrix D and an invertible matrix P such that:

$$
A = PDP^{-1}
$$

where A is the original matrix. This transformation is significant because diagonal matrices are much simpler to work with, especially regarding matrix exponentiation and other operations.

What Makes a Matrix Diagonalizable?

Not all matrices can be diagonalized. A matrix A is diagonalizable if it has enough eigenvalues to form a basis of eigenvectors. The following conditions are important:

1. Eigenvalues: A matrix must have n linearly independent eigenvalues to be diagonalizable, where n is the size of the matrix.
2. Multiplicity: The algebraic multiplicity of each eigenvalue must equal its geometric multiplicity.
3. Characteristic Polynomial: The characteristic polynomial of A should have n roots, counting multiplicities.

Steps to Diagonalize a Matrix

To diagonalize a matrix, follow these steps:

1. Find Eigenvalues: Solve the characteristic polynomial $\det(A - \lambda I) = 0$ to find the eigenvalues λ.
2. Find Eigenvectors: For each eigenvalue, solve the system $(A - \lambda I)v = 0$ to find the corresponding eigenvectors.
3. Form Matrix P: Construct the matrix P using the eigenvectors as columns.
4. Form Matrix D: Construct the diagonal matrix D using the eigenvalues on the diagonal.

5. Verify: Check if $A = PDP^{-1}$.

Example from Exercise 7

Let’s apply these steps to an example matrix from exercise 7.

Given Matrix

Consider the matrix:

$$
A = \begin{pmatrix}
4 & 1 \\
2 & 3
\end{pmatrix}
$$

Step 1: Find Eigenvalues

First, we calculate the characteristic polynomial:

$$
\det(A - \lambda I) = \det\begin{pmatrix}
4 - \lambda & 1 \\
2 & 3 - \lambda
\end{pmatrix}
$$

Calculating the determinant:

$$
(4 - \lambda)(3 - \lambda) - (2)(1) = \lambda^2 - 7\lambda + 10 = 0
$$

Factoring gives:

$$
(\lambda - 5)(\lambda - 2) = 0
$$

Thus, the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 2$.

Step 2: Find Eigenvectors

For $\lambda_1 = 5$:

$$
(A - 5I)v = 0 \implies \begin{pmatrix}
-1 & 1 \\

2 & -2
\end{pmatrix}\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = 0
\]

This leads to the equation \(x_1 = x_2\). Therefore, the eigenvector corresponding to \(\lambda_1 = 5\) is:

\[
v_1 = \begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

For \(\lambda_2 = 2\):

\[
(A - 2I)v = 0 \implies \begin{pmatrix}
2 & 1 \\
2 & 1
\end{pmatrix}\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = 0
\]

This leads to the equation \(2x_1 + x_2 = 0\) or \(x_2 = -2x_1\). Therefore, the eigenvector corresponding to \(\lambda_2 = 2\) is:

\[
v_2 = \begin{pmatrix}
1 \\
-2
\end{pmatrix}
\]

Step 3: Construct Matrix \(P\)

The matrix \(P\) formed by eigenvectors is:

\[
P = \begin{pmatrix}
1 & 1 \\
1 & -2
\end{pmatrix}
\]

Step 4: Construct Matrix \(D\)

The diagonal matrix D with eigenvalues is:

$$
D = \begin{pmatrix}
5 & 0 \\
0 & 2
\end{pmatrix}
$$

Step 5: Verify

To verify, we compute P^{-1} and check $A = PDP^{-1}$.

Calculating P^{-1}:

$$
P^{-1} = \frac{1}{-3}\begin{pmatrix}
-2 & -1 \\
1 & 1
\end{pmatrix} = \begin{pmatrix}
\frac{2}{3} & \frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3}
\end{pmatrix}
$$

Now compute PDP^{-1}:

$$
PDP^{-1} = \begin{pmatrix}
1 & 1 \\
1 & -2
\end{pmatrix} \begin{pmatrix}
5 & 0 \\
0 & 2
\end{pmatrix} \begin{pmatrix}
\frac{2}{3} & \frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3}
\end{pmatrix}
$$

Calculating this yields:

$$
A = \begin{pmatrix}
4 & 1 \\
2 & 3
\end{pmatrix}
$$

This confirms that A is diagonalizable.

Example from Exercise 20

Let’s move to exercise 20 for another example.

Given Matrix

Consider the matrix:

$$
B = \begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix}
$$

Step 1: Find Eigenvalues

First, we calculate the characteristic polynomial:

$$
\det(B - \lambda I) = \det\begin{pmatrix}
1 - \lambda & 2 \\
0 & 1 - \lambda
\end{pmatrix}
$$

Calculating the determinant gives:

$$
(1 - \lambda)^2 = 0
$$

Thus, the eigenvalue is $\lambda = 1$ (with algebraic multiplicity 2).

Step 2: Find Eigenvectors

For $\lambda = 1$:

$$
(B - I)v = 0 \implies \begin{pmatrix}
0 & 2 \\
0 & 0
\end{pmatrix}\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = 0
$$

This leads to $2x_2 = 0$, which implies $x_2 = 0$. We can choose x_1 freely, giving us:

$$
v_1 = \begin{pmatrix}
1 \\
0
\end{pmatrix}
$$

To find a second linearly independent eigenvector, we can use the generalized eigenvector approach. We solve:

$$
(B - I)v_2 = v_1 \implies \begin{pmatrix}
0 & 2 \\
0 & 0
\end{pmatrix}\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix}
1 \\
0
\end{pmatrix}
$$

This leads to $2x_2 = 1$ or $x_2 = \frac{1}{2}$ and x_1 can be chosen as 0:

$$
v_2 = \begin{pmatrix}
0 \\
\frac{1}{2}
\end{pmatrix}
$$

Step 3: Construct P

The matrix P is now:

$$
P = \begin{pmatrix}
1 & 0 \\
0 & \frac{1}{2}
\end{pmatrix}
$$

Step 4: Construct D

The diagonal matrix D is:

\[
D = \begin{pmatrix}
1 & 0 \\
0 & 1

\end{pmatrix}
\]

Step 5: Verify

Since \(B\) has only one eigenvalue with insufficient eigenvectors, it cannot be diagonalized. Thus, \(B\) is not diagonalizable.

Conclusion